AD-A071 069 WASHINGTON UNIV ST LOUIS MO DEPT OF SYSTEMS SCIENCE --ETC F/G 9/2
THE CONSTRAINT METHOD FOR SOLID FINITE ELEMENTS.(U)
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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFOSR-TR- 79-0844	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): THE CONSTRAINT METHOD FOR SOLID FINITE ELEMENTS	5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): I. Norman Katz	8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3122
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Systems Science & Mathematics, Washington University, Saint Louis, Missouri 63130	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A3
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washinton, D.C. 20332	12. REPORT DATE: September 1977
	13. NUMBER OF PAGES: 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

LEVEL

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

21p.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DDC RECEIVED JUL 11 1979 C

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

409 076

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

With on exception, all finite element software systems have element libraries in which the approximation properties of elements are frozen. The user controls only the number and distribution of finite elements. The exception is an experimental software system, ~~developed at Washington University. This system~~, called COMET-X, which employs conforming elements based on complete polynomials of arbitrary order. The elements are hierarchic; i.e. the stiffness matrix of each element is embedded in the stiffness matrices of all higher order elements of the same kind. The user controls not only the number and (continued)

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20. Abstract (continued)

distribution of finite elements but their approximation properties as well. This convergence can be achieved on fixed mesh. This provides for very efficient and highly accurate approximation and a new method for computing stress intansity factors in linear eleastic fracture mechanics. The theoretical developments are outlined, numerical examples are given and the concept of an advanced self-adaptive finite element software system is presented.

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The Constraint Method for Solid Finite Elements

Annual Technical Report, October 1, 1976 - September 30, 1977

by

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2. CURRENT AND ANTICIPATED RESULTS ANTICIPATED BY THE END OF INITIAL GRANT PERIOD

The ultimate objective of our research is to provide algorithms, and experimental digital computer programs which implement the constraint method for three dimensional analysis, and to demonstrate the effectiveness of the method in three dimensions. Since the approaches to be used in deriving these algorithms are based on similar approaches for two dimensional analyses, the first part of the initial grant period was spent in completing the formulation of algorithms in two dimensions so as to have a firm foundation for the more complex three dimensional work. Also in response to a suggestion by Dr. V. B. Venkayya of the Air Force Flight Dynamics Laboratory at Wright-Patterson AFB, the plate bending element of the Constraint Method is being developed further. The results obtained and/or anticipated for two dimensional analysis may be subdivided into three classes:

1). Results which apply to problems requiring only C° continuity (C° displacement fields, problems in plane elasticity)

2). Results which apply to problems requiring both C° and C^{1} continuity (coupled C° and C^{1} displacement fields)

3). Results which apply to problems requiring only C^{1} continuity (C^{1} displacement fields, plate bending problems)

2.1 C° Displacement Fields (plane elasticity)

It has been shown in [6, 7, 8, 9] that elemental arrays may be efficiently generated through the use of "precomputed" arrays - - that is arrays which are computed once, stored on permanent file, and then reused in all subsequent applications of the program. The new work done by the principal investigator and his collaborators has two objectives: the first is to show how the hierarchal C° elements (described in 1.2) for a quadratic functional may be formulated using precomputed arrays thus yielding a finite element technique which is especially suited to problems with local rapid variation of the function to be approximated. In particular, formulas for two-dimensional (hierarchal) element arrays for arbitrary polynomial order are derived, based on precomputed arrays. The second objective is to apply the combined approach of hierarchal elements and precomputed arrays to decide if a computed result has "converged". A common practice in finite element analysis is to solve a problem several times using successively required meshes i.e. to apply the procedure for h-convergence. If successive analyses agree then it is usually assumed that the finite element approximation is accurate. This procedure can be computationally expensive when several highly refined meshes are used. An alternative procedure is to use p-convergence, which, as pointed out in 1.1, has a faster rate of convergence to the true displacements. The computational effectiveness of the p-convergence procedure is demonstrated numerically using hierarchal elements and precomputed arrays.

Detailed formulas are given for calculation of stiffness matrices and for calculation of polynomial coefficients from nodal variables. Hierarchal nodal variables are presented together with some of the favorable consequences of using hierarchal nodal variables. Computation times for stiffness matrices are given in terms of equivalent time units (e.t.u.) for different methods. In Table 1 average Central Processor Unit (CPU) times are given for computing element stiffness matrices for several different problem types and polynomial orders p. For comparison purposes the computation time for an eight degree of freedom isoparametric quadrilateral using the general structural analysis program SAPIV of Bathe et al [10] has been included in the table. The time required to generate this 8×8 matrix is seen to be comparable to that required for the 45×45 hierarchal matrix of the torsion problem. This means that the constraint method allows many more degrees of freedom for the same computer cost.

Various numerical examples are analyzed in detail. One such example is a Cantilever Beam with Uniform Load, which we now describe.

2.1.1 An Example: Cantilever Beam with Uniform Load

It is known that in finite element problems where roundoff error is significant, an increase in the polynomial order may be advantageous [11]. This suggests that employing hierarchal elements with p-convergence may make it possible to use "poorly-shaped" elements - that is, elements of such proportions as would lead to ill-conditioned equations if low order polynomials were used. To test this idea, we consider the plane stress cantilever beam shown in Fig. 4, loaded with the edge stresses

$$\sigma_{yy} = \frac{\sigma_0}{3} \left(\frac{d}{L}\right)^2, \text{ for } 0 \leq x \leq L,\ y = d/2,$$

$$\sigma_{xx} = \frac{2\sigma_0 y}{d} \text{ and } \tau_{xy} = \frac{\sigma_0 d}{2L} \left[1 - \left(\frac{2y}{d}\right)^2\right], \text{ for } x = 0,\ -d/2 \leq y \leq d/2.$$

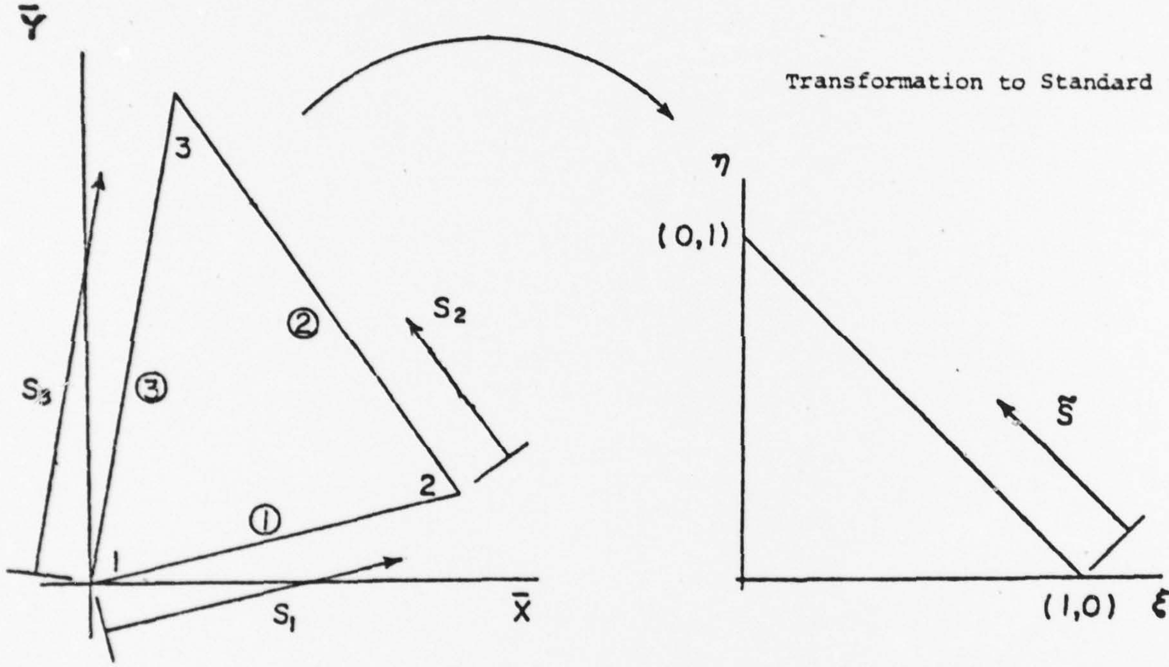
The beam is constrained against rigid body motion through the boundary conditions $u = v = 0$ for $x = y = 0$, and $u = 0$ for $x = 0$, $y = -d/2$. The five-element mesh employed in the analysis is shown in Fig. 4. Table 2 contains the displacements (in inches) at point A and the potential energy π calculated for the case of a deep beam ($L = 2.0$ in., $d = 1.0$ in.) with elastic modulus of 2.8×10^6 psi, and a slender beam ($L = 1.5$ in., $d = 0.1$ in.) with elastic modulus of 2.8×10^4 psi. Poisson's ratio was taken as 0.3 and the thickness as unity. The hierarchal elements exhibit convergence in both cases, even though the ratio of height to width is 1:30 for two of the elements in the mesh for the slender beam. Some numerical difficulty did occur, however, when an attempt was made to solve the slender beam problem for values of $L = 15$ in. and $d = 1.0$ in. Since a simple scaling of these dimensions led to accurate results, it may be that the difficulty was not due to numerical ill-conditioning, but instead was caused by the technique used to impose the boundary conditions - that is, the artificial stiffness coefficients introduced on the diagonal of the global stiffness matrix were not large enough.

For comparison purposes, Table 2 also contains values calculated by Mason and Case [12] using a fine mesh of low order elements. Since no analytical solution exists for the beam under the given loading, these authors validated their finite element model by applying it to a slightly different problem - involving the same beam, but a different loading - for which an analytical solution exists. The finite element and theoretical results for the displacements were found to agree to within 0.4%. In the present example, the results from [12] and from the hierarchal element solution were also found to agree to within 0.4%. The hierarchal element results may be considered closer to the exact values, if the good convergence of the computed values observed in Table 2 can be taken as an indicator of accuracy.

Table 3 contains CPU times for the equation solver for the hierarchal element approach.

2.2 Coupled C^0 and C^1 displacemnt fields

In this work the results of Kratochvil et al in [6] are generalized to problems with three independent displacement fields. An essential aspect of this approach is to transform a triangular element T in the x-y plane into a standard triangle $\tilde{T}$ with vertices at the origin and at a unit distance along the horizontal and vertical axes. Such a transformation is shown below where ξ and η represent coordinates in the plane of the standard triangle and $\bar{x}$ and $\bar{y}$ are local coordinates for the element with the first vertex of the element coinciding with the origin. The other two vertices and also the three edges are numbered in counter-clockwise order as shown. The justification for using the standard triangle is that integrations and matrix inversions are performed with respect to the standard triangle. Thus



Transformation to Standard triangle

they need be done only once and the results are stored and then used in all future applications of the program. Computation of the element stiffness matrix is thus reduced to computing a linear combination of a small number of precomputed matrices followed by pre- and post-multiplication by block diagonal matrices. The number of precomputed matrices which must be stored are considerably reduced by choosing hierarchal nodal variables.

2.3 C^1 displacement fields (plate bending problems)

It was suggested by Dr. V. B. Venkayya (of the Analysis and Optimization Group, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base) in a letter dated 12 December 1975 that further development of plate bending elements would be useful in order to realize the full potential of the constraint method. Accordingly, a sophisticated plate bending element, incorporating a complete $p^{\underline{th}}$ order polynomial with $p \geq 5$ and corrective rational functions is now being programmed and tested. We now describe some of this work.

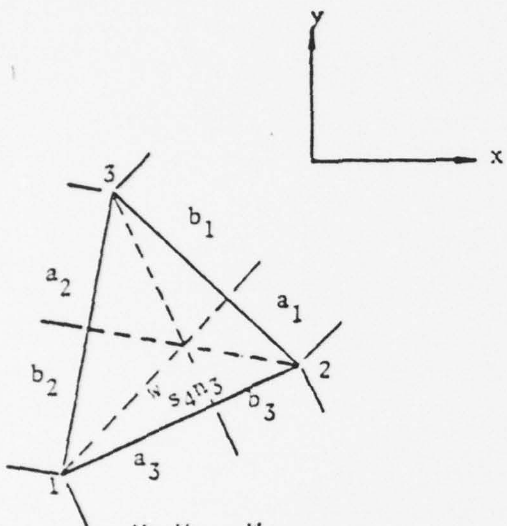
It is well known (see [13], for example) that exactly conforming (even at vertices) C^1 displacement fields cannot be formed merely by freely assembling finite elements. There are certain additional constraint equations which must be satisfied at vertices. The simplest form of these constraint equations has been given by Peano in [14], where a specially devised assembly procedure is presented which automatically enforces the constraint equations. Alternative methods for enforcing exact conformity when using displacement fields of arbitrary polynomial order p are:

a). creation of super-elements (or macro-elements) of arbitrary order $p \geq 5$. In these super-elements constraints are satisfied within each super-element leaving nodal variables on the boundary to be freely assembled [14].

b). Use of penalty functions to enforce constraints [14]

c). Supplementing $p^{\underline{th}}$ order polynomials with newly constructed corrective rational functions. This destroys the analytic character of the approximation at the vertices but permits free assembly of elements without enforcing constraints [14, 15].

The last alternative is the one which seems most promising. An algorithm has been developed for a C^1 (exactly) conforming triangular element which contains complete polynomials of order $p \geq 5$ and rational corrective functions. A typical element, of order $p = 5$, together with nodal variables is shown below.



$$\mu_1 = \frac{a_1 - b_1}{a_1 + b_1}$$

$$\mu_2 = \frac{a_2 - b_2}{a_2 + b_2}$$

$$\mu_3 = \frac{a_3 - b_3}{a_3 + b_3}$$

$w, w_{s_3}, w_{s_2},$

$w_{s_3 2}, w_{s_2 2}, w_{s_3 n_2}, w_{s_2 n_2}$

For the quintic C^1 element there are 24 independent nodal variables. The shape functions for each nodal variables have been given explicitly in [14, 15] in terms of triangular coordinates. The shape functions corresponding to second order tangential-normal derivatives at the vertices are the rational functions given below; all other shape functions are polynomials.

Nodal Variable	Shape Function
$w_{s_3 n_3}^{(1)}$	$\eta_1 = \dfrac{L_1^2 L_2^2 L_3}{L_2 + L_3}$
$-w_{s_2 n_2}^{(1)}$	$\eta_2 = \dfrac{L_1^2 L_2 L_3^2}{L_2 + L_3}$
$w_{s_1 n_1}^{(2)}$	$\eta_3 = \dfrac{L_1 L_2^2 L_3^2}{L_1 + L_3}$
$-w_{s_3 n_3}^{(2)}$	$\eta_4 = \dfrac{L_1^2 L_2^2 L_3}{L_1 + L_3}$
$w_{s_1 n_1}^{(3)}$	$\eta_5 = \dfrac{L_1 L_2^2 L_3^2}{L_1 + L_2}$
$-w_{s_2 n_2}^{(3)}$	$\eta_6 = \dfrac{L_1^2 L_2 L_3^2}{L_1 + L_2}$

(1)

It is important to observe that although rational functions are used in the basis, all terms which appear in the elemental stiffness matrix can be integrated explicitly without recourse to numerical quadrature. This was proved in [16]. An algorithm based on a hierarchical family of C^1 elements using corrective rational functions has been programmed and is now being tested on numerical examples.

In addition work is proceeding on the development of three dimensional finite elements for use in the constraint method as follows.

2.4 Nodal Variables and Shape functions for a hierarchical family of solid tetrahedronal elements

A table of basis functions for a triangular (two dimensional) hierarchical family was given in [14]. This is now being generalized to a table of basis functions for a tetrahedronal (three dimensional) hierarchical family. From this new table it will be possible to generate shape functions for solid C^0 elements and (if needed) corresponding nodal variables.

2.5 Papers prepared for publication and presentation at conferences

The work described in 2.1-2.4 will be reported to the professional community in the following papers to be submitted for publication in journals.

1). "Hierarchal Finite Elements and Precomputed Arrays", by Mark P. Rossow and I. Norman Katz, (to be submitted to Int. J. for Num. Method in Engr.).

2). "Nodal Variables for Conforming Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz and Mark P. Rossow, (to be submitted to Computers and Mathematics, with Applications).

3). "A family of C^1 triangular elements containing complete polynomials of arbitrary order, for application to problems in plate bending", I. Norman Katz, Barna A. Szabo and Olive Liu, (in preparation).

The following papers have been accepted for presentation at conferences.

4). "Hierarchical Approximation in Finite Element Analysis", by I. Norman Katz, International Symposium on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23 - 27, 1977.

5). "Efficient Generation of Hierarchal Finite Elements Through the Use of Precomputed Arrays", by M. P. Rossow and I. N. Katz, Second Annual ASCE Engineering Mechanics Division Specialty Conference, North Carolina State University, Raleigh, NC May 23 - 25, 1977.

6). "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I. Norman Katz, SIAM 1977 National Meeting, Philadelphia, PA, June 13 -15, 1977.

Table 1

Computation Time for Element Stiffness Matrices

Problem Type	Fields	p	Size of Matrices	CPU Time in e.t.u.
Plane stress (energy method)	Two displacements	5	42×42	2.1
Torsion (energy method)	Prandtl stress function	8	45×45	1.3
Plane stress (least squares)	Two displacements and three stresses	3	50×50	3.4
Plane stress (energy method) SAPIV [10]	Two displacements	1[a]	8×8	1.2
Plate bending (energy method) Ref. 7	One displacement	5[b]	18×18	0.9[c]

a Bilinear

b Incomplete fifth order polynomial

c Matrix multiplication routine used in defining an e.t.u. was not specified

Table 2

Results for Cantilever Beam Problem

		L =2, d=1		L=1.5, d=.1	
p	DOF	$v_A \cdot 10^8$	$\pi \cdot 10^8$	$v_A \cdot 10^6$	$\pi \cdot 10^4$
3	59	-8.683437	-3.6164721	-4.027224	-1.7574485
4	99	-8.757509	-3.6227009	-4.033644	-1.8010797
5	149	-8.760231	-3.6227475	-4.033650	-1.8010797
6	209	-8.764954	-3.6227711	-4.033656	-1.8010798
7	279	-8.764951	-3.6227734		
8	359	-8.765809	-3.6227744		
1152 Bilinear elements Ref. 12	2447	-8.7494	-	-	-
2160 Bilinear elements Ref. 12	4703	-	-	-4.0187	-

Table 3

CPU Time Required to Solve Equations
for Deep Cantilever Beam Problem

p	N_i	CPU Time in e.t.u.
3	62	20
4	102	36
5	152	56
6	212	96
7	282	174
8	362	298

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List of Professional Personnel Associated with the Research Effort

1. I. Norman Katz, Professor of Applied Mathematics & Systems Science, Washington University, St. Louis, Mo 63130

2. Barno A Szabo, A. P. Greenfelder Professor of Civil Engineering, Washington University, St. Louis, MO 63130

3. Mark P. Rossow, Assistant Professor of Civil Engineering, Washington University, St. Louis, MO 63130

4. Olive Y. Liu, Graduate Student, Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130

INTERACTIONS

On September 8, 1977, the Principal Investigator, I. Norman Katz, jointly with Barna A. Szabo presented a seminar on their current results at the Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base. The title of the talk was:

"Advanced Stress Analysis Technology"

An abstract of the talk is enclosed with this report. Forty to fifty people attended the seminar.

The principal contact at Wright-Patterson was Dr. V. N. Venkayya, Aerospace Engineer, Analysis and Optimization Branch, Structures Division. Other people with whom we spoke personally were: Dr. N. S. Khot, Dr. L. Berke, Dr. D. W. Quinn and Mr. N. D. Wolf of the Analysis and Optimization Branch; Dr. J. Gallagher and Mr. R. M. Bader of the Structural Integrity Branch.

Air Force Flight Dynamics Laboratory
Wright-Patterson Air Force Base
September 8, 1977

ADVANCED STRESS ANALYSIS TECHNOLOGY

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and

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ABSTRACT

With one exception, all finite element software systems have element libraries in which the approximation properties of elements are frozen. The user controls only the number and distribution of finite elements. The exception is an experimental software system, developed at Washington University. This system, called COMET-X, employs conforming elements based on complete polynomials of arbitrary order. The elements are hierarchic, i.e. the stiffness matrix of each element is embedded in the stiffness matrices of all higher order elements of the same kind. The user controls not only the number and distribution of finite elements but their approximation properties as well. Thus convergence can be achieved on fixed mesh. This provides for very efficient and highly accurate approximation and a new method for computing stress intensity factors in linear elastic fracture mechanics. The theoretical developments are outlined, numerical examples are given and the concept of an advanced self-adaptive finite element software system is presented.